Math 155, ***Lecture Notes- Bonds*** **Name_____________**

Section 9.4 *Comparisons of Series*

In this section we will still be studying series with positive terms. We will need to use our knowledge of the convergence and divergence characteristics of geometric series, harmonic series, *p*-series, and telescoping series to understand the convergence and divergence characteristics of series that do not fall within these categories. In general, we will be looking for series that are algebraically similar to the series we want to study, and we will try to understand convergence based on a comparison of two series, one of which has known convergence and divergence properties.

THEOREM 9.12 Direct Comparison Test

Let $0 < a_n \le b_n$ for all n.

1. If $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

2. If $\sum_{n=1}^{\infty} a_n$ diverges, then $\sum_{n=1}^{\infty} b_n$ diverges.

1. If the "larger" series converges, then the "smaller" series converges.
2. If the "smaller series diverges, then the "larger" series diverges.

Ex. 1: Determine the convergence or divergence of the series: $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$

"Think" $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1} \approx \sum_{n=1}^{\infty} \frac{1}{4n^{1/3}} \approx \frac{1}{4}\sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ ← This series diverges as a p-series, with $p = 1/3$.

Let's show divergence for the series, $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$, according to the Direct Comparison Test.

Let $\sum_{n=1}^{\infty} \frac{1}{4n^{1/3}} = \sum_{n=1}^{\infty} a_n$, with $\frac{1}{4n^{1/3}} = a_n$, and let $\sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1} = \sum_{n=1}^{\infty} b_n$, with $\frac{1}{4\sqrt[3]{n}-1} = b_n$. We need to show $0 < a_n \le b_n$ for all $n \ge 1$.

We know $1 > 0$ and $4n^{1/3} > 0$ for all $n \ge 1$. We can see that $\frac{1}{4n^{1/3}} > 0$ since a ratio of positive numbers is positive. (ROPNIP)

More Ex. 1:

This means that $a_n > 0$ for all $n \geq 1$.

We know $1 > 0$ and $4\sqrt[3]{n} - 1 > 0$ for all $n \geq 1$. We can see that $\frac{1}{4\sqrt[3]{n}-1} > 0$ since a ratio of positive numbers is positive. (RoPNiP)

This means that $b_n > 0$ for all $n \geq 1$.

We know $4\sqrt[3]{n} - 1 \leq 4\sqrt[3]{n}$ for all $n \geq 1$. We can use algebra:

$$\frac{1}{(4\sqrt[3]{n}-1)(4\sqrt[3]{n})} \cdot \frac{(4\sqrt[3]{n}-1)}{1} \leq \frac{1}{(4\sqrt[3]{n}-1)(4\sqrt[3]{n})} \cdot \frac{(4\sqrt[3]{n})}{1}$$

$$\frac{1}{4\sqrt[3]{n}} \leq \frac{1}{4\sqrt[3]{n}-1}$$, which means

$$a_n \leq b_n \quad \text{for all } n \geq 1.$$

We know $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{4n^{1/3}}$

$= \frac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^{1/3}}$ is a divergent p-series with $p = \frac{1}{3}$.

According to the Direct Comparison Test, $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{4\sqrt[3]{n}-1}$ is also a divergent series. □

Ex. 2: Determine the convergence or divergence of the series: $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$

"Think" $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}} \approx \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} \approx \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ ← This series converges as a p-series, with $p = \frac{3}{2} > 1$

Let's show convergence for the series, $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$, according to the Direct Comparison Test.

Let $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}}$, with $\frac{1}{\sqrt{n^3}} = b_n$, and let $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}} = \sum_{n=1}^{\infty} a_n$, with $\frac{1}{\sqrt{n^3+1}} = a_n$. We need to show $0 < a_n \leq b_n$ for all $n \geq 1$.

We know $1 > 0$ and $\sqrt{n^3+1} > 0$ for all $n \geq 1$. We can see that $\frac{1}{\sqrt{n^3+1}} > 0$ since a ratio of positive numbers is positive. (ROPNIP)

This means that $a_n > 0$ for all $n \geq 1$.

We know $1 > 0$ and $\sqrt{n^3} > 0$ for all $n \geq 1$. We can see that $\frac{1}{\sqrt{n^3}} > 0$ since a ratio of positive number is positive. (ROPNIP)

This means that $b_n > 0$ for all $n \geq 1$.

We know $\sqrt{n^3} \leq \sqrt{n^3+1}$ for all $n \geq 1$. We can use algebra:

$$\frac{1}{(\sqrt{n^3})(\sqrt{n^3+1})} \cdot \left(\frac{\sqrt{n^3}}{1}\right) \leq \frac{1}{(\sqrt{n^3})(\sqrt{n^3+1})} \cdot \left(\frac{\sqrt{n^3+1}}{1}\right)$$

$$\frac{1}{\sqrt{n^3+1}} \leq \frac{1}{\sqrt{n^3}}$$, which means

$a_n \leq b_n$ for all $n \geq 1$.

We know $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3}} = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ is a convergent p-series, with $p = 3/2$. According to the Direct Comparison Test, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ is also a convergent series. $\square$

THEOREM 9.13 Limit Comparison Test

Suppose that $a_n > 0$, $b_n > 0$, and

$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = L$$

where L is *finite and positive*. Then the two series $\Sigma\, a_n$ and $\Sigma\, b_n$ either both converge or both diverge.

Ex. 3: Determine the convergence or divergence of the series: $\displaystyle\sum_{n=1}^{\infty}\frac{5n-3}{n^2-2n+5}$

"Think" $\displaystyle\sum_{n=1}^{\infty}\frac{5n-3}{n^2-2n+5} \approx \sum_{n=1}^{\infty}\frac{5n}{n^2} \approx 5\sum_{n=1}^{\infty}\frac{1}{n}$ ← This series is a divergent harmonic series.

Let's show divergence for the series, $\displaystyle\sum_{n=1}^{\infty}\frac{5n-3}{n^2-2n+5}$, according to the Limit Comparison Test.

Let $\displaystyle\sum_{n=1}^{\infty}\frac{5}{n} = \sum_{n=1}^{\infty} b_n$, with $\frac{5}{n} = b_n$, and let $\displaystyle\sum_{n=1}^{\infty}\frac{5n-3}{n^2-2n+5} = \sum_{n=1}^{\infty} a_n$, with $\frac{5n-3}{n^2-2n+5} = a_n$. We need to show $a_n > 0$ and $b_n > 0$.

We know $5>0$ and $n>0$ for all $n \geq 1$. We can see that $\frac{5}{n} > 0$ since a ratio of positive numbers is positive. (RoPNiP) This means that $b_n > 0$ for all $n \geq 1$.

We know $5n-3>0$ and $n^2-2n+5>0$ for all $n\geq 1$. We can see that $\frac{5n-3}{n^2-2n+5} > 0$ since a ratio of positive numbers is positive. (RoPNiP) This means that $a_n > 0$ for all $n \geq 1$.

Consider $\displaystyle\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left[\frac{\frac{5n-3}{n^2-2n+5}}{\frac{5}{n}}\right]$

$$= \lim_{n\to\infty}\left[\frac{5n-3}{n^2-2n+5}\right]\cdot\left[\frac{n}{5}\right]$$

$$= \lim_{n\to\infty}\frac{5n^2-3n}{5n^2-10n+25}$$

More Ex. 3:

$$= \lim_{n\to\infty} \left[\frac{\frac{5n^2-3n}{1}}{\frac{5n^2-10n+25}{1}} \right] \cdot \left[\frac{\frac{1}{n^2}}{\frac{1}{n^2}} \right]$$

$$= \lim_{n\to\infty} \frac{5 - \frac{3}{n}}{5 - \frac{10}{n} + \frac{25}{n^2}}$$

$$= \frac{5-0}{5-0+0}$$

$$= 1$$

Since this limit is finite and positive, and since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{5}{n} = 5 \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent harmonic series, according to the Limit Comparison Test, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{5n-3}{n^2-2n+5}$ is also a divergent series. $\square$

Ex. 4: Determine the convergence or divergence of the series: $\sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$

"Think" $\sum_{n=1}^{\infty} \tan(\frac{1}{n})$ looks like $\sum_{n=1}^{\infty} \frac{1}{n}$. After looking the plot of the sequence of partial sums for $\sum_{n=1}^{\infty} \tan(\frac{1}{n})$, we don't see asymptotic behavior using "sum(seq(tan(1/K), K, 1, n))" in Ti-83 graphing calculator.

Let's show divergence for the series $\sum_{n=1}^{\infty} \tan(\frac{1}{n})$ since $\sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent harmonic series.

Let $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$, with $b_n = \frac{1}{n}$, and let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \tan(\frac{1}{n})$, with $a_n = \tan(\frac{1}{n})$. We need to show $a_n > 0$ and $b_n > 0$ for all $n \geq 1$.

We know $1 > 0$ and $n > 0$ for all $n \geq 1$, we can see that $\frac{1}{n} > 0$ since a ratio of positive numbers is positive. (RoPNiP) This means that $b_n > 0$ for all $n \geq 1$.

We know $\tan(x) > 0$ for all $0 < x < \frac{\pi}{2}$, by considering the graph of $y = \tan(x)$. We can see that $\tan(\frac{1}{n}) > 0$ since $0 < \frac{1}{n} < \frac{\pi}{2}$ for all $n \geq 1$. This means that $a_n > 0$ for all $n \geq 1$.

Consider $\lim_{n\to\infty} \left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty} \left[\frac{\tan(\frac{1}{n})}{\frac{1}{n}}\right] \leftarrow \frac{\tan(\frac{1}{\infty})}{\frac{1}{\infty}} = \frac{\tan(0)}{0} = \boxed{\frac{0}{0}}$ stop!

Indeterminate Form

Change to x's to use L'Hôpital's Rule

$$= \lim_{x\to\infty} \left[\frac{\tan(\frac{1}{x})}{\frac{1}{x}}\right]$$

$$= \lim_{x\to\infty} \frac{\frac{d}{dx}\left[\tan(\frac{1}{x})\right]}{\frac{d}{dx}\left[\frac{1}{x}\right]}$$

$$= \lim_{x\to\infty} \frac{\sec^2(\frac{1}{x}) \cdot (-1 \cdot x^{-2})}{(-1 \cdot x^{-2})}$$

More Ex. 4:

$$= \lim_{x\to\infty} \sec^2\left(\frac{1}{x}\right)$$

$$= \sec^2\left(\lim_{x\to\infty} \frac{1}{x}\right)$$

$$= \sec^2(0)$$

$$= 1$$

since this limit is finite and positive, and since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{1}{n}$ is a divergent harmonic series, according to the Limit Comparison Test, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \tan\left(\frac{1}{n}\right)$ is also a divergent series. $\square$

Ex. 5: Determine the convergence or divergence of the series: $\sum_{n=1}^{\infty} \frac{2}{3^n - 5}$

"Think" $\sum_{n=1}^{\infty} \frac{2}{3^n-5} \approx \sum_{n=1}^{\infty} \frac{2}{3^n} \approx 2\sum_{n=1}^{\infty} \frac{1}{3^n} \approx 2\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$

↑ This series is a convergent geometric series with $r=\frac{1}{3}$.

Let's show convergence for this series, $\sum_{n=1}^{\infty} \frac{2}{3^n-5}$, according to the Limit Comparison Test.

Let $\sum_{n=1}^{\infty} \frac{2}{3^n} = \sum_{n=1}^{\infty} b_n$, with $b_n = \frac{2}{3^n}$, and let $\sum_{n=1}^{\infty} \frac{2}{3^n-5} = \sum_{n=1}^{\infty} a_n$, with $a_n = \frac{2}{3^n-5}$.

We need to show $a_n > 0$ and $b_n > 0$.

We know $2 > 0$ and $3^n > 0$ for all $n \geq 1$. We can see that $\frac{2}{3^n} > 0$ since a ratio of positive numbers is positive. (RoPNiP) This means that $b_n > 0$ for all $n \geq 1$.

We know $2 > 0$ and $3^n - 5 > 0$ for all $n \geq 2^*$. We can see that $\frac{2}{3^n-5} > 0$ for all $\underline{n \geq 2}^*$ since a ratio of positive numbers is positive. (RoPNiP) This means that $a_n > 0$ for all $\underline{n \geq 2}^*$.

Consider
$$\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left[\frac{\frac{2}{3^n-5}}{\frac{2}{3^n}}\right]$$
$$= \lim_{n\to\infty}\left(\frac{2}{3^n-5}\right)\cdot\left(\frac{3^n}{2}\right)$$
$$= \lim_{n\to\infty} \frac{3^n}{3^n-5}$$
$$= \lim_{n\to\infty}\left[\frac{\frac{3^n}{1}}{\frac{3^n-5}{1}}\right]\cdot\left[\frac{\frac{1}{3^n}}{\frac{1}{3^n}}\right]$$
$$= \lim_{n\to\infty} \frac{1}{1-\frac{5}{3^n}}$$
$$= \frac{1}{1-0}$$
$$= \underline{\underline{1}}$$

More Ex. 5:

Since this limit, $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=1$, is finite and positive, and since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{2}{3^n} = 2\sum_{n=1}^{\infty}\left(\frac{1}{3}\right)^n$ is a convergent geometric series with $r=\frac{1}{3}$, according to the Limit Comparison Test, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{2}{3^n-5}$ is also a convergent series. $\square$

We'll look at this series using two different techniques.

Ex. 6: Determine the convergence or divergence of the series: $\sum_{n=1}^{\infty}\left[\frac{1}{n+1}-\frac{1}{n+2}\right]$

First, let's use some algebra to simplify the general term. $\sum_{n=1}^{\infty}\left[\frac{1}{n+1}-\frac{1}{n+2}\right] = \sum_{n=1}^{\infty}\left[\left(\frac{1}{n+1}\right)\left(\frac{n+2}{n+2}\right)-\left(\frac{1}{n+2}\right)\left(\frac{n+1}{n+1}\right)\right]$

$$= \sum_{n=1}^{\infty}\frac{n+2-n-1}{(n+1)(n+2)}$$

$$\boxed{= \sum_{n=1}^{\infty}\frac{1}{n^2+3n+2}}$$

"Think" $\sum_{n=1}^{\infty}\frac{1}{n^2+3n+2} \approx \sum_{n=1}^{\infty}\frac{1}{n^2}$ ← This series is a convergent p-series with $p=2$.

Let's show convergence for this series.

Let $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$, with $b_n = \frac{1}{n^2}$, and let $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{1}{n^2+3n+2}$, with $a_n = \frac{1}{n^2+3n+2}$. We need to show $a_n > 0$ and $b_n > 0$ for all $n \geq 1$.

We know $1>0$ and $n^2>0$ for all $n\geq 1$. We can see that $\frac{1}{n^2}>0$ since a ratio of positive numbers is positive. (RoPNiP) This means that $b_n>0$ for all $n\geq 1$.

We know $1>0$ and $n^2+3n+2>0$ for all $n\geq 1$. We can see that $\frac{1}{n^2+3n+2}>0$ since a ratio of positive numbers is positive (RoPNiP). This means $a_n > 0$ for all $n \geq 1$.

Consider $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right) = \lim_{n\to\infty}\left[\frac{\frac{1}{n^2+3n+2}}{\frac{1}{n^2}}\right]$

$$= \lim_{n\to\infty}\left[\frac{1}{n^2+3n+2}\right]\cdot\left[\frac{n^2}{1}\right]$$

$$= \lim_{n\to\infty}\frac{n^2}{n^2+3n+2}$$

$$= \lim_{n\to\infty}\left[\frac{\frac{n^2}{1}}{\frac{n^2+3n+2}{1}}\right]\cdot\left[\frac{\frac{1}{n^2}}{\frac{1}{n^2}}\right]$$

$$= \lim_{n\to\infty}\frac{1}{1+\frac{3}{n}+\frac{2}{n^2}}$$

More Ex. 6:

$$= \frac{1}{1+0+0}$$

$$= \underline{\underline{1}}$$ since this limit, $\lim_{n\to\infty}\left(\frac{a_n}{b_n}\right)=1$, is finite and positive, and since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty}\frac{1}{n^2}$ is a convergent p-series with $p=2$, according to the Limit Comparison Test,
$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty}\frac{1}{n^2+3n+2} = \sum_{n=1}^{\infty}\left[\frac{1}{n+1}-\frac{1}{n+2}\right]$$ also converges.

$\square$

Second Technique:

$\sum_{n=1}^{\infty}\left[\frac{1}{n+1}-\frac{1}{n+2}\right]$ looks like telescoping series.

Consider the partial sum for this series:

$$S_n = \sum_{k=1}^{n}\left[\frac{1}{k+1}-\frac{1}{k+2}\right]$$

$$S_n = \underbrace{\left(\frac{1}{2}-\frac{1}{3}\right)}_{k=1}+\underbrace{\left(\frac{1}{3}-\frac{1}{4}\right)}_{k=2}+\underbrace{\left(\frac{1}{4}-\frac{1}{5}\right)}_{k=3}+\underbrace{\left(\frac{1}{5}-\frac{1}{6}\right)}_{k=4}+\dots+\underbrace{\left(\frac{1}{n}-\frac{1}{n+1}\right)}_{k=n-1}+\underbrace{\left(\frac{1}{n+1}-\frac{1}{n+2}\right)}_{k=n}$$

$$S_n = \frac{1}{2}-\frac{1}{n+2}$$

Now, we can evaluate the limit of the partial sum.

$$S = \lim_{n\to\infty} S_n$$

$$S = \lim_{n\to\infty}\left(\frac{1}{2}-\frac{1}{n+2}\right)$$

$$S = \frac{1}{2}.$$ Since $S = \sum_{n=1}^{\infty}\left[\frac{1}{n+1}-\frac{1}{n+2}\right]$ and $\frac{1}{2} = \sum_{n=1}^{\infty}\left[\frac{1}{n+1}-\frac{1}{n+2}\right]$, we can see that the telescoping series is convergent and that its sum is $\frac{1}{2}$.